Our best juniors still struggle with Gauss's Law: Characterizing their difficulties

Rachel E. Pepper, Stephanie V. Chasteen, Steven J. Pollock and Katherine K. Perkins

Science Education Initiative and Department of Physics, University of Colorado, Boulder, CO 80309

Abstract. We discuss student conceptual difficulties with Gauss's law observed in an upper-division Electricity and Magnetism (E&M) course. Difficulties at this level have been described in previous work; we present further quantitative and qualitative evidence that upper-division students still struggle with Gauss's law. This evidence is drawn from analysis of upper-division E&M conceptual post-tests, traditional exams, and formal student interviews. Examples of student difficulties include difficulty with the inverse nature of the problem, difficulty articulating complete symmetry arguments, and trouble recognizing that in situations without sufficient symmetry it is impossible (rather than "difficult") to calculate the electric field using Gauss's law. One possible explanation for some of these conceptual difficulties is that even students at the upper level may struggle to connect mathematical expressions to physical meanings.

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1. INTRODUCTION

Gauss's law, $\oint \mathbf{E}\cdot\mathbf{dA} = Q_{enc}/\varepsilon_0$, is commonly taught in both introductory and upper-level physics courses. It has been shown through student interviews and a diagnostic exam that in introductory physics courses students have a number of problems with Gauss's law [1]. Similar difficulties have also been observed at the upper-level, based on a diagnostic exam, but these upper-division difficulties were not probed further with interviews [1].

Anecdotally, we find most instructors do not expect that juniors taking an advanced course in electricity and magnetism will have significant difficulties with Gauss's law. However, we show that even the best juniors still struggle with aspects of Gauss's law using evidence from the Colorado Upper-Division Electrostatics (CUE) diagnostic [2], exam questions, and student interviews.

In order to use Gauss' s law as a technique to solve for the electric field the student must be able to distinguish between problems that can and cannot be solved this way. For Gauss's law to be useful one must (1) be able to determine from the symmetry of the charge distribution what direction $\mathbf{E}$ points and on what variables $\mathbf{E}$ depends so that one can (2) create a Gaussian surface on which $\mathbf{E}\cdot\mathbf{dA}$ is known to be either constant or zero. Once such a Gaussian surface has been created, one can then (3) solve for $\mathbf{E}$ by pulling it out of the integral.

2. QUANTITATIVE EVIDENCE

Singh presents evidence that both introductory and upper-level students do poorly on a Gauss's law diagnostic while graduate students score much higher [1]. Both introductory and upper-level students score 49% post-instruction, though the sample size of upper-level students is much smaller (N= 28).

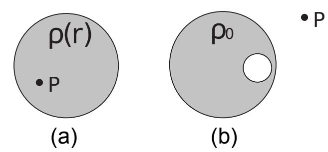

Figure 1. Illustrations from CUE diagnostic questions. Students are asked **not** to solve the question, but to give "the easiest method you would use to solve the problem" (half credit) and "why you chose that method (half credit)." **(a)** "A solid non-conducting sphere, centered on the origin, with a non-uniform charge density that depends on the distance from the origin, $\rho(r) = \rho_0 e^{-r^2/a^2}$. Find E (or V) at point P." **(b)** "A charged insulating solid sphere of radius R with a uniform volume charge ρ_0, with an off-center spherical cavity carved out of it. Find E (or V) at point P, at a distance $4R$ from the sphere."

We also see evidence of some students struggling with Gauss's law in the results of the CUE diagnostic, which we have given for the last 6 semesters at the University of Colorado (CU) as well as in several semesters at other institutions. Of the 325 total students who took the CUE after an E&M1 course, 33% did not recognize a radially symmetric problem as most easily solved with Gauss's law (see Fig. 1a for full problem). Of the 77% who correctly identified Gauss's law as the easiest solution technique, the average score for *explanation* of why and how Gauss's law was used was 59%.

In a second CUE question, involving a problem without sufficient symmetry for Gauss's law to be directly useful, students make the opposite mistake, and misidentify direct application of Gauss's law as an appropriate technique (see Fig. 1b for full problem). Of 325 students, 24% incorrectly chose Gauss's law as the easiest technique for this problem.

Students also revealed difficulties with Gauss's law on a midterm exam question asked in three different semesters of E&M1 at CU. This question asked

> Suppose I evenly fill a cube (length L on a side) with electric charges. Then imagine a larger, closed cubical surface neatly surrounding this cube (length $2L$ on a side). A) Is Gauss's law TRUE in this situation? (Briefly, why or why not?) B) Can one use Gauss's law to simply compute the value of the E-field at arbitrary points outside this charged cube? (Don't try, just tell me if you could, and why/why not?)

In a recent semester of a transformed course [3], most students correctly answered "yes" to the first part of the question, with an average score for that part of 89%. However, on the second part of the question students scored, on average, only 46% (this includes points for answer and explanation), and 31% of students received no points for their answer to this part of the question. Some common student responses will be discussed in Sections 3.1 and 3.3.

3. EVIDENCE FROM INTERVIEWS

In order to understand why upper-level students struggle with Gauss's law we conducted video-taped student interviews where students were asked to solve E&M problems in a think-aloud protocol. Interviews were somewhat open-ended and the interviewer asked follow-up or clarification questions. Four students who had completed an E&M1 course in the previous semester were asked a series of questions about Gauss's law.

Below we discuss several difficulties with Gauss's law observed during the course of these interviews. Due to the small sample size of students interviewed (who all got A's and B's in E&M1), we do not attempt to generalize to the larger population of upper-division E&M students.

3.1. Incorrect inferences about E based on the flux and an inverse problem

Two of the four students interviewed had the same difficulty when addressing the problem of an unevenly-shaped insulator of uniform charge density, ρ. These students incorrectly inferred from Gauss's law that the electric field at any point on such a Gaussian surface was determined only by the charge enclosed. Both students also did not clearly distinguish between the electric field at a single point on the surface and the flux through the entire surface, which may be partially leading to their struggle to correctly apply Gauss's law.

The first student claimed that

> ... the only thing that determines the flux out of it [the Gaussian surface] is the total enclosed charge. And so, if we took a sphere, and filled it completely with a completely uniform charge density in a spherical shape [draws an isolated solid circle with arrow's pointing outwards and spaced uniformly – see the left side of Fig. 2], say, then we could get a flux coming out of that area... 'cause then it would be uniform.

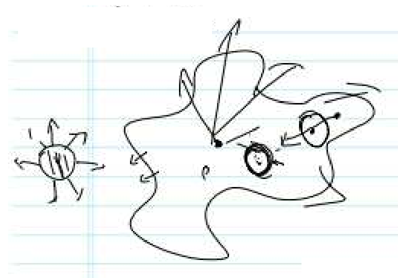

Figure 2. Student drawing from Gauss's law interview.

When asked if the electric field would be the same for the spherical Gaussian surface drawn inside the blob of insulator as for the isolated sphere she had drawn during her explanation, this student claimed "Yes. Because Gauss's law shows us that only the enclosed charge, um, matters." She proceeded to explain, that this was the case because "E-field lines only start and end on charges" so the lines from a point charge external to the Gaussian surface will pass through "so the only contributions that matter are the enclosed... charge contributions." It seems that she has inappropriately inferred that, because the flux depends only on the enclosed charge, the electric field will be uniform and radial on any spherical surface inside any insulator with uniform charge density ρ. Here she is also not specific about whether these contributions are to the flux through the Gaussian surface or to the electric field at an arbitrary point on the Gaussian surface.

Another student brought up the situation of an uneven shape of constant ρ on his own when asked to give examples of situations where Gauss's law is and is not useful for finding **E**. He used the uneven shape as an example where you could use Gauss's law to find the electric field. He explained that "the E-field... that passes through a Gaussian surface is only dependent on the Q enclosed." He then uses this statement to justify that one can find **E** using Gauss's law for a Gaussian surface inside the shape:

> On the inside, once again if it's [ρ is] constant, then that's fine, because there's... because it doesn't matter what the shape is looking like 'cause we're not looking on the outside. We're only looking... it's only dependent on the Q enclosed.

Here the student is not clear on what the "it" is that is only dependent on the Q enclosed – he could be thinking (correctly) about the flux, or (incorrectly) about the electric field. This student later states that he thinks that **E** is the same throughout a Gaussian surface inside this object, but expresses some discomfort with his understanding, saying "'Cause if there's Q on the outside, the charge, you know, is making an E-field as well... and therefore it must affect the **E** field at that point [points to a point on the Gaussian surface] as well. So I'm still... I'm still not really happy with Gauss's law."

The difficulties these students have making incorrect inferences about the electric field from Gauss's law's statement about flux through a surface are similar to difficulties discussed by Singh [1], and by Wallace and Chasteen in regards to Ampere's law [4]. In both cases, the authors describe students incorrectly inferring that because the integral is zero, the field in the integrand is zero as well. The students we observe may also be using reasoning in which students consider the right hand side of an equation the cause and the left hand side its effect [5] – in this case thinking that Q_{enc} is the only cause of the electric field on the Gaussian surface.

Our observations of these students discussing Gauss's law show that advanced students may make incorrect inferences from the integral to the integrand in more general circumstances. It is notable that the two students discussed above were top students in their E&M course; both received above 95% for their overall course score. The problem of confusing the electric field and flux, or making incorrect inferences in the context of an inverse problem where **E** cannot be solved for algebraically persists even among the best upper-level students.

We also occasionally observe a more basic problem with the inverse nature of Gauss's law. As is common for introductory students [1, 6], a few upper-level students use Gauss's law in a rote way by just solving $EA = Q_{enc}/\varepsilon_0$ without considering symmetry or visualizing the electric field. This type of solution was seen in one interview, and in work for solutions to the exam problem involving a cube, described in Section 2.

3.2. Difficulty with symmetry arguments

In order to create a proper gaussian surface, one must use the symmetry of the problem to determine what direction the electric field points and on which variables it depends. Many of the students interviewed could make these predictions in highly symmetric situations, but could not justify one or both of the direction or the dependence of the electric field.

A common expert-like argument to justify the direction and dependence of the electric field relies only on the geometry of the charge distribution. One student in a previous set of interviews makes such an argument when describing the electric field around a long charged cylinder with $\rho = Ks$, where s is the radial cylindrical coordinate: "If you rotate the cylinder, the E-field should be the same. So there can't be any ϕ dependence," and "if you go up and down it looks the same" so there can't be any z dependence. This argument can be extended to include the direction of the electric field by arguing that an infinite cylinder looks the same looking in $+\hat{z}$ as in $-\hat{z}$, so that an electric field pointing in $\hat{z}$ would be contradictory. An analogous argument can be made to eliminate a $\hat{\phi}$ component.

A second type of symmetry argument, based on superposition and Coloumb's law, can be employed to deduce the direction of the electric field. For instance, when considering the electric field at an arbitrary point above an infinite line charge, one can imagine that for every small piece of the line to the right of the point of interest, there is another small piece of the line at the same distance from the point on the left, and that when the electric field from these two pieces are added at the point of interest, the horizontal components cancel, leaving only a radial component. In Griffiths' upper-division E&M textbook [7], this type of superposition symmetry argument is commonly used; it is made directly in a worked example for an infinite line charge, and students are often asked in homework to use this same technique to solve for the electric field on the symmetry axis of rings, squares, and disks of charge [7, p. 62]. On the other hand, Griffiths models a geometry-based symmetry argument only once [7, p. 70] in the context of Gauss's law (and this in a footnote) and the other Gauss's law examples simply state the direction with statements like "by symmetry" or "symmetry dictates" [7, p. 73].

Perhaps it should not be surprising, then, that students almost exclusively make superposition symmetry arguments, even when these arguments are unproductive. All four students who were asked about an infinite line of charge discussed only the lack of a z-component and made superposition symmetry arguments. When asked directly about why there was no z-dependence of the electric field in this situation, three of the students used a superposition symmetry argument that horizontal components of **E** cancel and leave only z components, despite the fact that they had been asked about dependence, not direction. One of these students also tried to use a similar superposition argument to explain why the electric field points radially outward from a sphere. While this argument is possible, the student did not succeed, and from an expert's perspective, it is easier to make this argument based on geometry rather than superposition of electric fields. It seems that superposition symmetry arguments are the predominant type of symmetry arguments students use.

Students also used superposition arguments when not applicable. For instance, one student discussed the use of Gauss's law near an unevenly-shaped insulator with uniform ρ. He drew a Gaussian surface close to the surface of the object with the same shape as the object, and was trying to decide whether it was possible to use Gauss's law on that surface. He explains that he is trying to think what the electric field looks like by mentally adding up the contributions from the different parts of the shape. This is a difficult task, and the student ended up incorrectly deciding that the electric field was perpendicular to the Gaussian surface and uniform on the Gaussian surface. It is possible that students misapplying superposition arguments, rather than employing geometrical symmetry arguments are leading to student difficulties applying Gauss's law in novel situations.

Intriguingly, none of the students interviewed made a complete argument for both the dependence and the direction of the electric field – perhaps because completely determining both is difficult without employing some geometry-based symmetry arguments.

Manogue *et al.* point out in the context of Ampère's law that even immediately after explicit instruction in expert-like geometrical symmetry arguments, students struggle to recreate them when solving a new problem [8]. It seems that these sort of arguments are difficult and non-intuitive even for upper-division students.

We occasionally also observe other student difficulties related to symmetry, such as non-expert definitions of symmetry. For instance, one student seemed to use the word symmetry to mean that her Gaussian surface had a "uniform" electric field (i.e. that the field was constant on the surface), and even said at one point that "it's [the electric field is] symmetric coming out through this circular part of the cylinder," whereas we think her intended meaning was that the electric field was constant on that surface.

3.3. Belief that the solution is messy rather than impossible

Another issue we see at the upper-level is students believing that it will be complex, "difficult," or "messy" to use Gauss's law to find the electric field of a charge distribution, when it is actually impossible. For instance, in the exam question described in Section 2, 36% of students in a recent semester of our E&M course (N=59) said it would be difficult but possible to use Gauss's law to solve for the electric field of a uniformly charged cube. In actuality we cannot determine the direction and dependence of the electric field without resorting to direct integration of Coulomb's law. We therefore cannot create a Gaussian surface on which we know **E** can be removed from the integral.

In our interviews discussing a similar problem from Singh's survey [1, Appendix B, #22] all four students made this mistake. Students are asked to choose all the Gaussian surfaces which can be used to determine the electric field at a point near an infinite uniform line charge. While the students recognized a cylinder as the easiest option, they also said that the spherical and cubic Gaussian surfaces shown would be difficult but possible. For instance, one student said, referring to using the spherical Gaussian surface to find the electric field: "I would have to think some more. Maybe do some trig identities...figure it out. It would be a little more complicated [than the cylinder], but we could figure that out."

It could be that students are not yet familiar with solving inverse problems (i.e. where **E** cannot be simply solved for algebraically), so do not realize that they are sometimes not possible to solve. It could also be that students just are not thinking through the problem, and that once they tried to solve for the electric field, they would realize that the problem is impossible. However, in interviews, this did not seem to be the case; rather, students required much prompting from the interviewer before reaching this conclusion. It could also be that students at the upper division have seen so many fancy tricks to solve seemingly-impossible problems that they have some faith that all problems are solvable if one just knows the right trick.

4. CONCLUSION

While conceptual difficulties with Gauss's law are often thought of as an issue only for introductory students, we show that a complete understanding of Gauss's law is still lacking for many juniors, including some top students. Upper-level students interviewed make incorrect inferences about the electric field based on Gauss's law, are unclear in distinguishing flux and electric field, struggle to articulate complete symmetry arguments, and believe using Gauss's law will be difficult rather than impossible in some situations. In upper-level courses it may be helpful to provide instruction that explicitly addressees these conceptual pitfalls

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